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Geometrical features of crack network formation on rock surfaces during cooling

Abstract. This study aimed to determine the most probable geometric shape of cracking cells based on the principle of minimising the energy required to cause failure in the surface layer of rocks. The research methodology was based on analysing the classical isoperimetric problem. This states that, among shapes with the same perimeter, the circle has the largest area, while among polygons, the regular *n*-gon has the largest area. The energetically most favourable configuration of the crack system was substantiated, taking into account the physical conditions governing crack formation. The theoretical propositions were then compared with experimental data and the results of previous thermomechanical rock failure studies. It was found that cooling heated surfaces predominantly results in a polygonal crack structure with hexagonal cells forming. It was demonstrated that the hexagon is the most optimal geometrical figure, in that it allows for dense packing on a plane without the formation of additional voids or stresses, and at the same time provides the minimum perimeter for a given area. The size of the cracking cells was found to be determined by a combination of factors, the most important of which were the physical and mechanical properties of the rock, the magnitude of the temperature gradient and the cooling regime. The characteristic cell size decreases as the temperature difference between heating and cooling increases. The broad

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spectrum of structural dimensions formed, ranging from fractions of a millimetre to several metres, was confirmed. The practical significance of these results lies in their potential application in predicting fracture parameters and controlling rock failure processes during mineral deposit development, and in designing technologies involving thermal effects on rock masses

Keywords: thermal cyclic loading; isoperimetric problem; regular tessellation; honeycomb structure; thermal cyclic failure

Introduction

The successful execution of contemporary geotechnological operations is contingent on a capacity to regulate the development of fractures within rock formations. This applies to the development of geothermal systems and enhanced oil recovery and mineral extraction, as well as to ensuring the stability of mine workings and underground structures. It is particularly interesting to look at thermal cyclic loading processes that follow a "heating-cooling" sequence. This is because these processes cause significant temperature differences in the surface layers of rocks. This leads to the development of tensile stresses and the formation of ordered polygonal systems of surface cracks. The geometrical characteristics of these systems directly influence the subsequent failure of rock, its permeability, the intensity of defect development and the energy required for mechanical comminution. The formation of polygonal crack structures is governed by regularities. This makes it both a fundamental scientific problem in fracture mechanics and an important applied task. Solving it will contribute to the development of more efficient technologies. These technologies are for the thermal treatment of rock masses.

The review by L.N. Wong & Z. Liu (2025) provides a summary of the current state of research, systematically presenting the results of numerous experimental and numerical studies of thermal shock in rocks, and analysing its effects on changes to the structure, mechanical properties and durability of rock masses. The authors showed that thermal failure is determined by the complex interplay of temperature regimes, heating and cooling rates, mineral composition, texture, initial fracturing and boundary conditions. However, they also concluded that there is currently no universal mechanical model capable of explaining the regularities of thermal rock failure under different loading conditions. The authors placed particular emphasis on the need to develop new theoretical approaches. These approaches should account for the spatial organisation of crack systems. They should also account for the characteristics of the actual rock structure.

T. Chen *et al.* (2022) made a significant advance in the study of polygonal crack geometry by numerically reproducing the process of crack formation on rock surfaces during cooling. The authors demonstrated that the type of boundary conditions dictates the nature of crack formation, while the thickness of the surface layer and structural heterogeneity regulate the average size, regularity, and morphology of the polygonal cells formed. Building on these findings, X. Hu *et al.* (2022) have significantly advanced the understanding of the mechanisms underlying crack propagation through the use of modern numerical methods. They demonstrated that crack propagation paths in granite are governed by mineral structural heterogeneity, the mechanical properties of individual grains, and the nature of intergranular contacts. Conversely, Q. Li *et al.* (2021) revealed that thermal damage evolves via the gradual accumulation of microcracks that form a spatially connected macroscopic failure system as thermal loading increases. Overall, the findings of these studies confirm that the final configuration of the crack network is the result of a complex interaction between the microstructural characteristics of the rock and the evolution of localised damage.

The effect of repeated thermal loading on the development of crack structures has been a particular focus of researchers. For example, D. Zhu *et al.* (2023) investigated how repeated heating and cooling cycles affect the tensile strength of granite. They experimentally demonstrated that as the number of thermal cycles increases, the density of microcracks and the degree of internal damage to the rock gradually increase, resulting in a corresponding reduction in tensile resistance. Additionally, they demonstrated that the failure pattern shifts from the formation of isolated defects to the development of an interconnected crack network, which ultimately results in a loss of load-bearing capacity. R.A. Haque *et al.* (2023) conducted a study that examined the evolution of polygonal crack systems in clay materials during repeated wetting and drying cycles. Based on experimental observations and statistical analysis, the authors concluded that the crack network gradually restructures geometrically over successive cycles, with short, unstable segments disappearing and polygons acquiring a more regular shape. The authors of the study concluded that the self-organisation of the polygonal network is the result of the system's tendency to move towards a more energetically favourable state.

T. Chen *et al.* (2026) investigated the regularities that govern the formation of polygonal cracks in multilayered rocks under cooling-induced shrinkage, using a three-dimensional thermomechanical model. The authors determined that the geometry of weak interlayers has a substantial impact on the density of the crack network, the dimensions of the polygonal blocks, and the redistribution of thermal stresses. Their findings confirm that the morphology of the polygonal structure is determined by both the temperature regime and the structural heterogeneity of the rock. The development of rock thermomechanics and geomechanics in Ukraine was given a sound scientific basis by fundamental studies carried out by Ukrainian researchers, which occupy an important place in the development of this field.

Ukrainian researchers D.L. Vasyliev *et al.* (2020) made a significant contribution to the investigation of the mechanisms governing the destruction of the surface layers of geomaterials under thermal shock, describing in detail the nature of rock-surface failure during rapid cooling. They demonstrated that critical tensile stress development causes degradation of the near-surface zone. However, the geometrical principles governing the regular packing of resulting crack contours require further analytical substantiation. A.F. Bulat *et al.*'s (2022) monographic study provides comprehensive substantiation for the application of nonlinear dynamic systems theory and fractal analysis to problems involving the destruction of geomaterials. They showed that the geometrical parameters and scale invariants of crack systems directly reflect the kinetics of energy dissipation within a loaded rock mass. The post-peak stability of rocks subjected to local external effects was investigated by S. Skipochka *et al.* (2025). The authors showed that failure development is determined by stress redistribution and the accumulation of potential energy within the rock mass. Mathematical modelling was also shown to make it possible to predict the loss of system stability. The findings demonstrate the efficacy of employing an energy-based method for analysing crack formation processes.

Analysis of the current research literature indicates that most studies focus on the experimental investigation of thermal damage to rocks and the assessment of changes to their physical and mechanical properties. Other areas of focus include modelling microcrack initiation and development and analysing the effects of temperature regimes, material heterogeneity and boundary conditions on the nature of failure. Significant progress has been made in reproducing crack propagation processes through numerical modelling, as well as in identifying the regularities that govern the evolution of polygonal networks. At the same time, the studies analysed give virtually no consideration to the fundamental geometrical principles. These principles underpin the self-organisation of polygonal crack systems. The studies also do not substantiate the reasons for the statistical predominance of hexagonal cells. They do not propose an analytical criterion either. This criterion would be capable of explaining the selection of their characteristic dimensions. These dimensions depend on the physical and mechanical properties of rocks. They also depend on the parameters of thermal loading. The application of the isoperimetric approach as a theoretical basis for describing the formation of polygonal structures during the rapid cooling of rocks remains largely unexplored. This study aimed to determine the most probable geometric shape of cracking cells by analysing the classical isoperimetric problem and the principle of minimising the energy required to fail the surface layer of rocks. It also aimed to derive analytical relationships for estimating the characteristic dimensions of resulting hexagonal structures depending on thermal cyclic loading conditions.

Materials and Methods

This theoretical and analytical study aimed to establish the regularities governing the formation of surface crack systems in rocks subjected to thermal cyclic loading. The study focused on the formation process of surface crack systems in rocks under thermal cyclic loading. The study examined the geometrical regularities that govern the formation of a polygonal surface crack structure, as well as the impact of the structural characteristics of rocks and thermal shock parameters on the shape and dimensions of cracking cells.

To achieve this, a combination of theoretical and analytical research methods was employed. These included the theoretical substantiation of the optimal geometrical form of a polygonal crack system. This was based on an analysis of the isoperimetric problem. Another was the determination of the geometrical cell shape. This was the one that ensures minimum energy expenditure during crack system formation. Then there was the analysis of the conditions for regular tessellation of a plane by regular polygons. Another was the generalisation of published experimental data. This was on the formation of polygonal crack systems in rocks. It was under different thermal cyclic loading regimes. Another was the assessment of the effects of rock structural heterogeneity. This was on thermal shock parameters. It was also on the shape and characteristic dimensions of polygonal cells. Finally, there was the comparison of the regularities governing polygonal structure formation in rocks. This was with analogous processes in other natural and biological systems.

When a rock that has been heated is cooled at its surface, multiaxial tensile thermal stresses arise within the surface layer. Once these stresses reach a critical level, the surface layer fails, forming a crack system that divides it into individual cells. These cells are predominantly regular or nearly regular polygons. Therefore, establishing the regularities governing the formation of their geometrical shape constitutes a distinct scientific problem. The following assumptions were made when analysing the formation of the polygonal crack system: fragmentation occurs along the path requiring the least energy; cracks generally propagate along straight lines; elastic fracture occurs, with plastic deformation being absent or insignificant; and the cells are packed as densely as possible over the surface, without empty spaces between them.

The energy required to form a single destruction cell will be considered minimal when, for a given cell area *S*, its perimeter is minimal. The essence of the classical isoperimetric problem, the focus of this study, can be summarised as follows: it is necessary to determine the figure with the largest area among all possible figures having the same perimeter. In other terms, given a set of geometrical configurations with an equal external contour length, the configuration providing the maximum area must be identified. This problem is of fundamental importance in geometric optimisation and has practical applications in analysing the regularities that govern the spatial distribution of cracks. In the present context, analysing the isoperimetric problem explains why a hexagonal, or honeycomb, crack structure predominantly forms as it is the most efficient configuration in terms of minimising surface energy and ensuring a uniform stress distribution. This explanation is in line with the geometrical crack patterns that are produced when materials fail due to heat and that have been observed in both natural conditions and in experimental studies.

The fundamental principles of energy minimisation that underlie the self-organisation of crack structures are based on the classical isoperimetric problem. As early as the works of the ancient Greek mathematician Zenodorus in around 200 BC, it was demonstrated that the circle has the largest area of any plane figure with a given perimeter (Heath, 1921). Nevertheless, even though this proposition may seem obvious, its mathematical proof is not simple. It was not until the eighteenth century that L. Euler (1744) proved the isoperimetric property of the circle using the calculus of variations and provided a geometric justification for the problem. In the present study, the isoperimetric approach was employed to substantiate the most probable geometry of a polygonal crack system formed during the thermal cracking of rock surfaces. The classical isoperimetric inequality was the basis of the solution to the isoperimetric problem. This inequality establishes a relationship between the length *L* of a closed curve, or polygon, and the area *S* of the surface enclosed by it. The isoperimetric inequality states that:

$$4\pi S \leqslant L^2. \tag{1}$$

This inequality becomes an equality if and only if the curve is a circle. This relationship was used as a theoretical criterion. It was used for evaluating the most energetically favourable shape of polygonal cells. It reflects the fundamental principle that, for a given area, the circle has the smallest perimeter. Since forming each crack requires additional energy, the most favourable shape for a polygonal cell is the one that minimises the total crack length. For quantitative comparisons of regular polygons, the isoperimetric quotient *Q* was used:

$$Q = \frac{4\pi S}{L^2}, \tag{2}$$

where *S* – is the area of the polygon; *L* – its perimeter. The isoperimetric quotient was used as the main criterion for comparing regular polygons quantitatively, according to how closely they approximate a circle (which corresponds to the maximum value of $Q = 1$. The greater the coefficient, the smaller the perimeter of a figure with the same area, and consequently the lower the potential energy expenditure required to form the polygonal crack system. For a regular *n*-gon, the isoperimetric quotient was determined using the following relationship:

$$Q_n = \frac{\pi}{n \cdot tg\frac{\pi}{n}}. \tag{3}$$

This formula was used to calculate the isoperimetric quotients of regular polygons with between three and ten sides. These values were then used in a comparative analysis of different geometric shapes, considering minimum perimeter length and the possibility of regular tessellation of a plane. The analysis also took into account the classical isoperimetric theorems, which state that among all *n*-gons with the same perimeter, the regular *n*-gon has the largest area and that, among *n*-gons inscribed in a given circle, the regular *n*-gon has the maximum area and perimeter. These propositions were used to provide a theoretical basis for the optimal geometry of polygonal cells. To estimate the thermal stresses arising in the surface layer of a rock under thermal shock and first-kind boundary conditions, the classical thermoelasticity equation was used:

$$\sigma_* = \frac{\beta \cdot E \cdot (T_i - T_c)}{1-\mu}, \quad (4)$$

where β – is the coefficient of linear thermal expansion of the rock; E – Young's modulus; T_i – is the initial rock temperature; T_c – is the temperature of the cooling medium; μ – is Poisson's ratio. Formula (4) was used to calculate the magnitude of the thermal stresses that arise in the surface layer of the rock during rapid cooling. The following relationship was used to estimate the characteristic size of the polygonal cells under specified cooling condition:

$$g = \frac{0.45 \cdot K^2 \cdot \sigma_c^2}{\pi^2 \cdot \sigma_*^2 \cdot \sigma_t^2}, \quad (5)$$

where K – is the cohesion modulus of the rock; σ_c – is the compressive strength; σ_t – the tensile strength. This formula (5) was used to analyse the impact of the physical and mechanical properties of rocks, as well as thermal cyclic loading regimes, on the size of the characteristic cells within the polygonal crack system. To verify the proposed theoretical model, a comparative analysis was conducted. This analysis was of published experimental studies. These studies concerned the formation of polygonal crack systems. These systems were formed under different thermal cyclic loading regimes. Laboratory experiments and field observations investigating the formation of polygonal structures over a broad range of scales, from fractions of a millimetre to several metres, were analysed. Additionally, the effect of the structural heterogeneity of rocks on the geometry of the polygonal crack system was analysed, as well as the effect of the temperature difference between heating and cooling on the characteristic dimensions of the polygonal cells.

In the final stage of the study, a comparative analysis was carried out of the patterns governing the formation of polygonal structures in rocks, in relation to similar processes in other natural and biological systems. This made it possible to assess the universality of the principle of energy minimisation as a general mechanism for the formation of cellular structures of various physical natures. The proposed model is based on the assumption of homogeneity and isotropy of the material within the study area, as well as on the sufficient uniformity of the temperature field in the zone where surface cracks form. At the same time, the model does not take into account natural fracturing, mineralogical heterogeneity, anisotropy of the physical and mechanical properties of rocks, moisture content, residual stresses, or local inhomogeneities in the temperature field. Consequently, it describes the general geometric patterns of polygonal fracture system formation and can be used as a theoretical basis for the further development of models of thermal rock failure.

Results

The theoretical analysis first determined which polygonal cell geometry is most energetically favourable when a surface crack system is formed. The underlying premise is that forming each new crack creates fracture surfaces, which requires energy. Therefore, all other conditions being equal, the most probable cell configuration should provide the minimum total boundary length for a given area. Analysis of experimental observations indicates that such cells predominantly take the form of regular or nearly regular polygons. The most probable geometric shape of these cells was determined by quantitatively analysing the relevant relationship. To quantify this relationship, the isoperimetric quotients Q of regular polygons with between three and ten sides were analysed using formula (3). The calculation results are presented in Table 1.

Table 1. *Isoperimetric quotients for different polygons*

Number of polygon sides, *n*	3	4	5	6	7	8	9	10
Q	0.61	0.78	0.86	0.90	0.95	0.96	0.97	0.98

Source: *calculated by the authors based on their own computations*

The isoperimetric quotient of regular polygons approaches unity as the number of sides increases, as a comparative analysis showed. Of the figures considered, the regular decagon has the highest value, with an isoperimetric quotient of around 0.98. This indicates the most favourable ratio of area to contour length, meaning that the specific energy expenditure required to form the polygon boundaries is lowest. From the standpoint of fracture mechanics, this relationship has important physical implications. The same area with a smaller perimeter will result in a shorter total length of newly formed crack surfaces. This means less energy is needed to form them. Thus, the isoperimetric quotient can be considered a geometrical indicator of the potential energy efficiency of a polygonal structure.

At the same time, the results of the calculations indicate that the maximum value of the isoperimetric coefficient does not, in itself, determine the geometry of a polygonal fracture system. Although polygons with a large number of sides are closer in shape to a circle and exhibit better isoperimetric properties, most of them are unable to regularly fill a plane without creating gaps or overlaps. Under these conditions, minimising the perimeter cannot be the sole criterion for the self-organisation of the system. Consequently, the results of the first stage of the study show that determining the most probable geometry of polygonal cells requires simultaneous consideration of two independent requirements: maximum approximation of the shape to the isoperimetric optimum and the possibility of regular tessellation of the plane. It is precisely the analysis of the combined effect of these criteria that enables the determination of the geometry that is realised during the thermal fracturing of rock.

The results of the isoperimetric analysis showed that the geometric efficiency of a regular polygon increases consistently with the number of its sides. However, this relationship alone cannot explain the formation of polygonal cracks in rocks during thermal failure. Another fundamental condition, derived from the physical nature of the process, must also be considered. A polygonal crack system develops as a continuous network that divides the surface layer of a rock into separate fragments without producing voids or overlaps, unlike an isolated geometrical figure. Therefore, the geometry of an individual cell must allow the entire plane to be regularly tiled, since only such a configuration is compatible with the continuous propagation of cracks within the surface layer. A comparative analysis of regular polygons revealed that only three geometric figures satisfy the condition of regular tessellation: the equilateral triangle, the square and the regular hexagon (Fig. 1). Furthermore, when a system of elastic circles is arranged on a plane and compressed, the circles transform specifically into hexagons (Fig. 2).

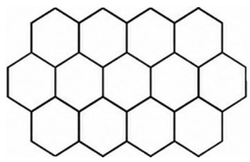

Figure 1. *Dense packing of hexagons on a flat surface*
Source: *compiled by the authors*

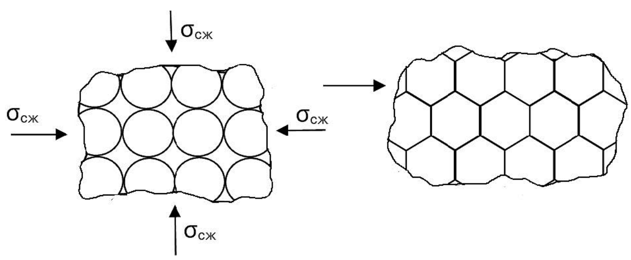

Figure 2. *Transformation of elastic circle s into hexagons under compression*
Source: *compiled by the authors*

Practical observations suggest that the resulting crack structure closely resembles a honeycomb pattern, meaning it is predominantly hexagonal. Both during intensive cooling and during temperature-induced polymorphic transformations of quartz, this relationship is observed. Heating the low-temperature modification of β-quartz to 573°C transforms it into the high-temperature modification, α-quartz. This transformation is diffusionless and occurs without breaking the principal interatomic bonds, accounting for its virtually instantaneous nature. Subsequent cooling below 573°C normally causes α-quartz to transform back into β-quartz, accompanied by volumetric changes in the crystal lattice and the development of internal stresses.

It has been established that when quartz that has been heated above 573°C is rapidly cooled, a specific system of inversion cracks forms within its structure. The geometry of this system is predominantly honeycomb, or hexagonal, and it can be detected using microscopic methods. Honeycomb cracks produced by cooling quartz rapidly from above the inversion range to below the inversion temperature are diagnostic $\alpha \rightarrow \beta$ of quartz phase transformations. These transition cracks only form under intensive cooling, such as with water, and are absent during slow cooling, such as in air. The degree of quartz cracking under these conditions is very high. In the experiments, a crack network with a characteristic size of 0.15 × (0.2 ÷ 0.4) mm was obtained. A comparison of the three possible tessellation patterns revealed significant differences in their geometrical efficiency. The isoperimetric quotient of an equilateral triangle is approximately 0.605; that of a square is 0.785; and that of a regular hexagon is 0.907. Therefore, among all regular polygons capable of tiling a plane in a regular manner, the hexagon provides the minimum total boundary length for the same cell area.

The formation of cracks is accompanied by an energy cost associated with the creation of new surfaces; therefore, the system seeks to minimise their total length. Given a regular tessellation of the plane, this problem boils down to finding a geometric compromise. If only the minimisation of the perimeter is considered, the optimal shape is a circle, which cannot tightly fill the plane. If only the tessellation condition is considered, the possible shapes are a triangle, a square and a hexagon. The simultaneous combination of these two independent criteria leads to a single analytical solution – a regular hexagon, which provides the smallest perimeter for a given cell area. This geometric principle forms the basis for the self-organisation of real fractured structures in rock formations.

The crack system evolves towards a hexagonal configuration in stages. During the initial stage of thermal shock, an isotropic field of biaxial tensile stresses forms on the surface of the rock. Due to the mineralogical heterogeneity of the material and the random distribution of local stress concentrators, individual cracks form independently of each other. Further propagation of the defects transforms the local stress state of the surface. Each newly formed crack produces a surrounding relaxation zone. This is also known as an unloading zone. The interaction between these zones via adjacent mechanical stress fields transforms the random crack formation process into a self-organising regime. As energy is dissipated, the number of possible network configurations becomes limited: polygons with a small number of sides (triangles) have an excessively high specific boundary length, which is energetically unfavourable, whilst polygons with $n > 6$ do not provide a continuous tessellation of the plane. As a result, the system stabilises at an optimal compromise solution – a regular hexagon. The structural and thermal heterogeneity of real rocks causes local deviations of individual cells from the ideal geometry, resulting in the formation of adjacent pentagonal and heptagonal contours; however, the overall tendency towards the formation of a honeycomb topology remains dominant.

Laboratory experiments involving the rapid cooling of granite, basalt and other brittle rock surfaces demonstrate the formation of such polygonal networks. The cells formed are predominantly pentagonal, hexagonal and heptagonal in shape, with a clear statistical predominance of hexagonal elements. Under natural conditions, this geometric mechanism manifests itself in the formation of columnar structures during the cooling of crystallised lava. A honeycomb system of cracks forms on its surface, with hexagonal blocks ranging in size from a few centimetres to 1.5 m, penetrating into the massif to a depth of up to 30 m. The mechanisms governing the development of such structures at great depths depend significantly on high geostatic pressure. The planetary and universal nature of this phenomenon is confirmed by the discovery of similar hexagonal cooling formations on the surface of Mars. Thus, a comparison of theoretical propositions with empirical and field data reveals their qualitative consistency. This confirms the adequacy of the proposed isoperimetric model for predicting the topology of thermal cracking in geomaterials.

Nevertheless, it should be noted that the verification conducted is qualitative in nature. The proposed model describes the fundamental geometrical regularities governing the formation of a polygonal network. However, quantitative prediction of the proportions of polygons of different shapes requires consideration of material heterogeneity, crack-development kinetics, the three-dimensional stress state, and other factors beyond the scope of the present study. The linear dimensions of polygonal cells are determined by the balance between the accumulation of elastic energy in the near-surface layer and its dissipation during failure, unlike their shape. Analysis of the developed model and formula (5) shows that the scale of the resulting network changes systematically under the influence of thermomechanical parameters. Tensile stresses are accelerated by an increase in the intensity of thermal loading, the elastic modulus, or the coefficient of linear thermal expansion. Consequently, the critical state is reached at a shorter distance between adjacent defects, resulting in a denser network with smaller individual cells. The tensile strength of the rock has the opposite effect. A higher concentration of thermal stresses is required to initiate failure when tensile strength increases. This leads to larger polygons being formed, with the spacing between cracks increasing.

The analytical relationships established provide a physical basis for the known experimental data for granites and basalts, where accelerated cooling is always accompanied by a refinement of the fracture structure. Thus, during thermal fracture, the geometric topology of the network is governed by the principles of energy optimisation, whilst its scale factor is a function of the properties of the geomaterial and the thermal loading conditions. Combining the isoperimetric criterion with the condition of regular tessellation of the plane analytically proves that a regular hexagon is the only energetically optimal configuration that ensures minimal energy expenditure for the formation of fracture surfaces. The transition from chaotic crack initiation to an ordered hexagonal network results from the self-organisation of the system through the interaction of local stress relaxation zones. The heterogeneity of the geomaterial's properties determines the statistical nature of the model, which is expressed in the dominance of hexagonal cells alongside the presence of defective pentagonal and heptagonal contours. A fundamental distinction between the factors governing crack formation has been established: the topology of the network is governed by geometric principles of perimeter minimisation, whilst its linear scale is a function of the thermomechanical parameters of the medium and the intensity of the heat flow.

Discussion

The results obtained allow to explain the predominantly hexagonal shape of the polygonal crack system in terms of minimising fracture energy and geometrically optimising the surface. Unlike traditional approaches, which focus on strength criteria and the mechanisms responsible for initiating thermal cracks, the proposed isoperimetric model explains the geometric nature of the resulting defect system. Recent mathematical studies, particularly the review by J. Li & S. Pan (2025), support the fundamental significance of the selected isoperimetric approach, as they demonstrated the universal applicability of isoperimetric inequalities in different types of spaces. Their proposition that the isoperimetric inequality is a key instrument for optimising geometrical forms appears well founded. However, unlike this general theoretical approach, the present study is restricted to tessellation problems, specifically the dense tiling of the Euclidean plane. This is critical for explaining the formation of a hexagonal crack network within a rock mass.

The results obtained are consistent with those of W. Nie *et al.* (2023), who used numerical modelling to demonstrate the formation of an extensive system of thermal cracks in rocks. While the authors of that study analysed the kinetics and mechanisms of crack development, this research focuses on explaining why hexagonal cell geometry predominates specifically during thermal cracking. These findings are also consistent with those of Q. Lu *et al.* (2023), who investigated thermal damage in rocks and shales under thermal shock conditions. They confirmed the intensive development of cracks and changes in rock mass permeability as a function of temperature gradients. The proposed approach reveals the topological characteristics of the resulting polygonal structure, unlike their studies, which focused on the quantitative assessment of accumulated damage.

The study by T. Wang *et al.* (2026) is interesting because it draws parallels with the present research. This earlier study analysed the influence of surface defects on crack initiation during soil desiccation. The authors support the proposition in this study that local stress concentrators on the surface trigger the formation of regular structures. Their conclusion that defects play a decisive role in crack initiation seems entirely justified. However, in contrast to their strategy, which centred on observing changes in soil deformation, this study aims to provide a detailed explanation of the shift from a random arrangement of defects to a hexagonal pattern in solid rocks. This transformation is driven by the principle of reducing isoperimetric energy. The results of this study are consistent with the observations of Y. Akiba *et al.* (2021a; 2021b), who used field measurements of columnar joints and solidified lava to identify universal statistical patterns in the geometry of polygonal cracks. Unlike their predominantly statistical and descriptive studies, the isoperimetric model proposed here provides a physical and geometrical explanation for the predominance of hexagonal cells.

Important confirmation of the hierarchical nature of the established model is provided by the results of C.-S. Tang & T. Wang (2026), who demonstrated that the architecture of crack networks is governed by a recursive stress-field feedback mechanism. They demonstrated that over 80% of secondary microcracks originate at the midpoint of a parent crack, and that mutual attraction occurs when the distance between them decreases to approximately two to four times their width, resulting in the formation of nodal junctions. This process leads to a shift from chaotic cracking to ordered polygonal structures. Unlike the objects investigated by C.-S. Tang & T. Wang (2026), in which this process develops gradually during soil desiccation, similar hexagonal cell topologies form in brittle rocks under conditions of intensive thermal shock. The characteristic hexagonal network pattern is therefore a universal outcome of geometrical self-organisation. This is governed by the isoperimetric principle of fracture-energy minimisation. This is irrespective of the nature and rate of the external influence.

In the context of energy optimisation principles governing cracking, the proposed isoperimetric approach closely aligns with recent specialist studies on damage evolution in rocks subjected to thermal effects. The evolution of rock damage during rapid cooling has been extensively examined by L. Dong *et al.* (2024), and the results of this contemporary experimental data confirm it. The authors examined the temperature dependence of the physical and mechanical properties of hot, dry rocks when subjected to a sudden reduction in temperature. They demonstrated through experimentation that a high cooling rate is a critical trigger for the widespread initiation of macrocracks and the rapid degradation of the rock mass. Their approach is different. It records the loss of strength. This is unlike the present study. It explains how this avalanche-like evolution of damage geometrically self-organises in space. This is done so as to minimise energy expenditure.

This degradation inevitably involves a complex transformation of the system's energy balance, which was analysed in depth by W. Shen *et al.* (2026). Their study considered energy evolution in granite. This was under cyclic loading combined with thermal shock and seawater cooling. The authors convincingly demonstrated that crack formation is the principal mechanism through which accumulated thermoelastic energy is dissipated in rocks. While the study by W. Shen *et al.* (2026) shows that a lot of energy is spent creating new surfaces, the isoperimetric approach proposed here explains why this dissipation is optimum through the formation of a hexagonal network.

Similar processes can be clearly represented in terms of their kinematics and spatial development using modern numerical modelling methods, as demonstrated in the study by L. Su *et al.* (2026). The authors used the discrete element method (DEM) to visualise the propagation of cracks and dynamic damage in granite that was directly subjected to high-temperature thermal shock. The model they developed shows how microcracks initially develop chaotically, gradually merging into larger structures as they are influenced by temperature variations. While DEM modelling shows the progressive course of failure over time, the analytical geometrical model in this study identifies the ideal topological outcome of this process: formation of a hexagonal cell, as this is the most energy-efficient geometric shape.

Analytical degradation models, such as that developed by Y. Li *et al.* (2026), also reflect the theoretical basis of these processes. This study proposed a comprehensive mathematical model of damage evolution in granite subjected to extreme high-temperature thermal shock. The researchers' primary focus was on quantitatively assessing the deterioration of the material's mechanical properties and changes in its structural integrity. This study builds on their work by translating quantitative damage indicators into the qualitative domain of topology, demonstrating that the final macrocrack configuration is strictly governed by the macroscopic isoperimetric principle.

In the context of geometrical optimisation, the study by Q. Ma *et al.* (2023), which directly applies the concept of planar tessellation, merits particular attention. The authors applied a two-scale tessellation approach within the discrete element method. This was to analyse the thermal failure of granite under different temperature conditions. However, in their study, polygonal subdivision was primarily used as a mathematical tool, or mesh, to accurately model a continuum numerically. In contrast, in the present study, tessellation is not treated as an artificial approximation tool, but rather as a fundamental physical and geometrical mechanism that explains the natural formation of regular polygons.

The geometry of cracking described in this study is in line with the key review by L. Wong & Z. Liu (2025), which identified thermal shock as a major cause of damage to underground structures. While that review documents the formation of polygonal structures as a typical consequence of energy dissipation, the present study provides a mathematical demonstration of why this self-organisation tends towards a regular hexagon under planar tessellation conditions. When analysing the geometrical parameters of crack networks, the recently published dataset by A. Asadian *et al.* (2026) should also be considered. This contains labelled geometrical characteristics for hundreds of shrinkage-crack networks. The authors recorded the predominance of polygonal structures. These had mainly triangular, quadrilateral, and hexagonal cell geometries. A. Asadian *et al.*'s (2026) conclusion that the formation of polygonal structures is statistically characteristic of failure processes appears well founded. However, in contrast to their method, which merely documented geometric parameters without elucidating the underlying physical mechanism, this study delves into the origin of these structures. Thus, the data of A. Asadian *et al.* (2026) provide an empirical basis for observation. This study explains why the hexagonal form becomes dominant during intensive cracking. This addresses a gap in the theoretical understanding of the mechanism responsible for the formation of such structures.

The methods developed by the authors for thermal cyclic weakening and volumetric treatment of strong rock masses, which are protected by Ukrainian Utility Model Patents No. 154853 (2023) and Ukrainiфan Utility Model Patent No. 161823 (2026), represent a practical development of the theoretical propositions obtained in this study. Unlike the present study, these patents primarily focus on the technological implementation of artificial fracture generation and substantiating thermal treatment regimes. By contrast, the present work reveals the underlying geometrical nature of the crack system formed when the patented methods are implemented. The proposed isoperimetric approach extends the existing understanding of thermomechanical rock failure and is of considerable importance for optimising geotechnologies, the effectiveness of which depends directly on the controlled management of rock mass permeability.

Conclusions

The mechanism governing the formation of a polygonal cooling-crack network has been theoretically substantiated. This was based on the isoperimetric approach and dense planar tessellation. It has been established that, of the shapes capable of regularly tiling a surface, the regular hexagon provides the greatest isoperimetric efficiency. The predominance of hexagonal geometry has been shown to result from minimising the total crack perimeter, which corresponds to the lowest energy expenditure required for failure. The statistical predominance of hexagonal polygonal cells has been shown to be a natural consequence of the geometrical self-organisation of the crack system. This system forms during the thermal failure of rocks.

The proposed model explains how individual pentagonal and heptagonal cells can form next to hexagons in the real world. This is associated with heterogeneity in mineral composition, local variations in stress levels and the natural anisotropy of rocks. The characteristic size of polygonal cells has been shown to be determined by the intensity of thermal loading and the physical and mechanical properties of the rock. As the temperature difference between the heated rock and the cooling medium increases, the density of cracks rises and the average size of the polygonal cells decreases systematically. The theoretical conclusions are in line with the experimental observations, which confirm the formation of a honeycomb-like polygonal structure with characteristic cell sizes ranging from fractions of a millimetre to several metres, depending on the thermal exposure conditions.

The proposed model's practical significance lies in its potential use for predicting the morphology of polygonal crack systems during thermal failure of rocks in thermal drilling, thermal treatment, and the stimulation of geothermal-reservoir permeability. It can also be used to interpret natural polygonal structures of different origins. Future research should extend the model to account for the anisotropy of the physical and mechanical properties of rocks, the heterogeneity of their mineral composition, the three-dimensional development of crack networks and integrate the analytical approach with modern numerical modelling methods. This would enable to progress from qualitative descriptions to quantitative predictions of the parameters of polygonal structures.

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Conflict of Interest

None.

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Геометричні особливості формування сітки тріщин на поверхні гірських порід при охолодженні

Анотація. Метою роботи було встановлення найімовірнішої геометричної форми комірок розтріскування на основі принципу мінімізації енергетичних витрат при руйнуванні поверхневого шару гірських порід. Методологічну основу дослідження становили аналіз класичної ізопериметричної задачі, згідно з якою серед фігур з однаковим периметром максимальна площа має коло, а серед багатокутників – правильний *n*-кутник. З урахуванням фізичних умов формування тріщин виконано обґрунтування найбільш енергетично вигідної конфігурації системи тріщин. Теоретичні положення зіставлено з експериментальними даними та результатами раніше проведених досліджень у галузі термомеханічного руйнування гірських порід. У результаті встановлено, що при охолодженні нагрітих поверхонь формується переважно полігональна структура тріщин з комірками шестикутної форми. Показано, що шестикутник є оптимальною геометричною фігурою, що забезпечує мінімальний периметр при заданій площі та можливість щільного пакування на площині без утворення додаткових порожнин або напруг. Встановлено, що розмір комірок розтріскування визначається сукупністю факторів, серед яких ключовими є фізико-механічні властивості породи, величина температурного градієнта та режим охолодження. При збільшенні різниці температур між нагріванням та охолодженням відбувається зменшення характерного розміру комірок. Підтверджено широкий діапазон розмірів структур, що формуються – від часток міліметра до метрів. Практичне значення результатів полягає у можливості їх використання для прогнозування параметрів тріщинуватості та управління процесами руйнування гірських порід при розробці родовищ, а також при проектуванні технологій термічного впливу на масив

Ключові слова: термоциклічне навантаження; ізопериметрична задача; регулярна теселяція; стільникова структура; термоциклічне руйнування